

Interval Scale as Group Generators

Tsubasa Tanaka

Tokyo University of the Arts
t-tsubasa@y4.dion.ne.jp

Kiyoshi Furukawa

Tokyo University of the Arts
kf@zkm.de

ABSTRACT

The purpose of this paper is to propose “interval scale,” a new concept defined as a set of ordered pitch-class intervals. Unlike an ordinary scale, this concept restricts usable intervals and doesn’t restrict pitch classes directly. This provides possibilities that interval scales can be used in atonal music that uses all pitch classes and can be used to express some differences in a similar way to ordinary scales, depending on the selection of the elements. In this paper, we first present two existing musical pieces that can be interpreted as being based on interval scales, and see the possible effectiveness of this concept to express some senses of tonality. Next, we show that an interval scale is a generating set of a mathematical group and prove the necessary and sufficient condition for an interval scale to generate all pitch classes as a condition of atonality. Furthermore, the relationship between tone row and interval scale is examined, and the necessary and sufficient condition for an interval scale to be preserved by several tone-row transformations is proved. These results will provide a basic understanding and some criteria of selecting interval scales for composers who create music based on this concept.

1. INTRODUCTION

In tuning systems that divide an octave into 12 notes like 12 equal temperament, pitch classes $C, C\#, \dots, B$ are identified with the set \mathbb{Z}_{12} that consists of residue classes of $\mathbb{Z} \bmod 12$. A scale in such tuning systems can be represented as a subset of \mathbb{Z}_{12} . For example, $\{\bar{0}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{9}, \bar{11}\}$ is the diatonic scale and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$ is the chromatic scale, where \bar{m} denotes the class represented by an integer m ($0 \leq m \leq 11$)¹. Similarly, tuning systems which divide an octave into n notes are represented as \mathbb{Z}_n , and scales in such tuning systems are subsets of \mathbb{Z}_n .

However, \mathbb{Z}_n can be interpreted in another way. The collection of all of ordered pitch-class intervals² in a tuning system of n notes can be also represented as \mathbb{Z}_n (i.e.,

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¹ In this paper, we identify a scale with another scale when they are identical as sets. For example, we ignore the difference between Ionian scale $\{\bar{0}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{9}, \bar{11}\}$ and Mixolydian scale $\{\bar{7}, \bar{9}, \bar{11}, \bar{0}, \bar{2}, \bar{4}, \bar{5}\}$.

² The interval between two pitch classes is called ordered pitch-class interval [1]. However, we use the term “interval,” if it would not cause confusion.

$= \{y - x \pmod n | x, y \in \mathbb{Z}\}$, where \mathbb{Z} represents pitches). The purpose of this paper is to investigate this interval system \mathbb{Z}_n and propose the concept of “Interval scale,” which is defined as follows:

Definition 1 (interval scale³). *An interval scale is a set of intervals represented as a subset of the interval system \mathbb{Z}_n (The empty set is not regarded as an interval scale).*

The reason we think this concept is important is that it may have the possibility to express some differences depending on the selection of its elements in a similar way to the selection of ordinary scales such as major scale, minor scale, church modes, etc. The advantage of the interval scale over an ordinary scale is that it can be used in atonal music that uses all pitch classes, as is explained later. In contrast, ordinary scales are not effective in atonal music because they restrict usable pitch classes.

In this paper, we describe the properties of interval scale in several aspects. In Section 2, we present two existing musical pieces that can be interpreted with different interval scales, and see the possible effectiveness of this concept to express some senses of tonality. However, objective evaluation of how different interval scales are perceived differently is beyond the scope of this paper. Section 3 and beyond focus on investigating the mathematical structures of interval scales. In Sections 3 and 4, we show that the interval scale has a deep relationship with group theory and examine the diversity of interval scales as group generators. In Section 5, the relationship between tone rows and interval scales is investigated.

2. ROLE OF INTERVAL SCALE

In this section, we give two examples of musical pieces to show that there are relevant existing pieces and to observe the role of interval scales.

2.1 Ligeti’s “Étude 2: Cordes à vide”

The first example, Ligeti’s piano piece “Étude 2: Cordes à vide (Open Strings),” was composed using many perfect fifths, as the title shows. Thanks to the extensive use of specific intervals, this piece has a strong sense of tonality.

The collection of all intervals between consecutive notes from the beginning to the end of bar 2 forms an interval scale $\{\bar{5}, \bar{6}, \bar{7}, \bar{8}\}$ (see Figure 1). This interval scale has a small number of elements and consists of consecutive integers. Not only $\bar{7}$ and $\bar{5}$, which represent ascending and

³ Although we use this name, it would also be appropriate to call it “mode of intervals” after Messiaen’s piece (or concept) “Mode de valeurs et d’intensités (Mode of values and intensities)” [2, 3]

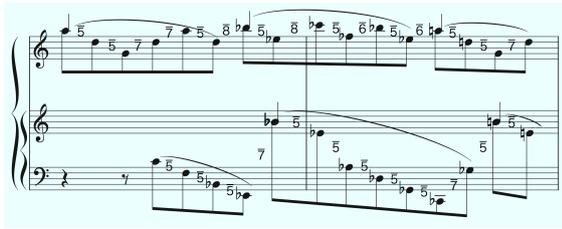


Figure 1. The opening of Ligeti’s “Étude 2: Cordes à vide.” The numbers denote intervals between consecutive notes in unit of semitone. [4]

descending perfect fifths, respectively, but also the neighboring intervals $\bar{6}$ and $\bar{8}$ are used in this fragment. This enables the melodies to deviate a little from the movement of fifths and provide variety to the melodic movements.

Concerning the constitution of pitch classes (scale), however, all of 12 pitch classes appear by the end of the second slur of the right hand part. Therefore, this fragment can be interpreted as being based on the chromatic scale.

However, the sonority of this piece is different from ordinary pieces that use twelve tones. This specificity would be well explained by the gap between the size of scale and that of interval scale. We can consider that the few elements in the interval scale is associated with the large bias of sonority, and that the interval scale plays a role to mediate atonality and a sense of tonality.

2.2 Berg’s “Violin Concerto”

The next example is the twelve-tone row of Berg’s “Violin Concerto” (Figure 2).



Figure 2. 12-tone row of Berg’s “Violin Concerto.” [5, 6]

The regularity of this tone row is easy to find. The major 3rd and minor 3rd appear early, and the last three intervals are two semitones. The collection of these intervals forms an interval scale $\{\bar{2}, \bar{3}, \bar{4}\}$. These intervals are closely related to tonal music. The intervals $\bar{3}$ and $\bar{4}$ are constituent of major and minor triads, and $\bar{2}$ is one of the most important melodic intervals in tonal music. In spite of his use of twelve-tone technique, Berg is famous for pieces that hold some sense of tonality [7]. These intervals would be a good explanation for the sense of tonality in this piece.

Although the sonority of this tone row (or this piece) and the selection of intervals are different from Ligeti’s example, these examples share common properties: (1) all of 12 pitch classes appear; (2) the interval scales have only a small number of elements; and (3) the elements in the interval scales are consecutive integers. Are these occurring accidentally or inevitably? These questions are discussed in the later chapters.

The tone row of this piece is a significant contrast to the “all-interval series,” which is a twelve-tone row that has all types of intervals except for $\bar{0}$. All-interval series can be interpreted that its interval scale is $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}\}$,

$\bar{11}\}$ and that the sense of tonality is excluded systematically by avoiding the bias of intervals as well as that of pitches. This contrast indicates that there is a freedom of selecting intervals in a tone row. We can consider that the use of an interval scale in a tone row is to organize the character of sonority by utilizing this freedom, and that the interval scale plays a role to coordinate atonality and a sense of tonality, which are seemingly contradictory properties.

3. ALGEBRAIC STRUCTURE

In the previous section, we presented existing musical pieces and showed their common characteristics and the possible effectiveness of interval scales. The next problem is how to select appropriate interval scales to compose new pieces. Before dealing with this problem, we examine the mathematical structure of the interval scales and \mathbb{Z}_n .

3.1 Additive Operation of Intervals

As is mentioned in Section 1, a tuning system like n -equal temperament and its interval system are $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$. \mathbb{Z}_n is not just a set, but it has an algebraic structure, a group in which the additive operation of two given elements is defined. This additive operation is naturally understood as the addition of intervals. For example, “ $\bar{2} + \bar{10} = \bar{0}$ ” means “two semitones + ten semitone equals an octave.” However, if the same addition is interpreted as an addition of two pitches, the meaning can not be understood naturally. This is the same with the time: Although we would think “2 o’clock plus 2 o’clock equals 4 o’clock” is meaningless, we would find meaning in “two hours plus two hours equals four hours.” Therefore, we find more interest in the intervals system \mathbb{Z}_n than in the tuning system \mathbb{Z}_n from the viewpoint of group theory.

3.2 Interval Scale as Group Generators

Let’s confirm the definition of group [8]:

Definition 2 (Group). *Let G be a set in which an operation “ \cdot ” for all $a, b \in G$ is defined and $a \cdot b$ is also contained in G . G is a group if it satisfies the following three axioms:*

1. *Associativity:* $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. *Identity element;* $\exists e \in G$ such that $\forall a \in G$ $ae = ea = a$.
3. *Inverse element:* $\forall a \in G, \exists b \in G$ such that $a \cdot b = b \cdot a = e$.

The interval system \mathbb{Z}_n with the additive operation of intervals “+” satisfies these axioms. $\bar{0}$ is the identity element and $\overline{n-a}$ is the inverse element of \bar{a} . This inverse element is denoted by $-\bar{a}$.

If H is a subset of a group G and it is also a group, H is called a subgroup of G . The number of elements of G is called the order of G . If a set G satisfies the definition except for the second and third conditions, G is called a semigroup [9]. A semigroup is not required to have an identity element and inverse element. If H is a subset of a semigroup G and it is also a semigroup, it is called a subsemigroup of G .

are cases where the group generated by an interval scale is smaller group than whole \mathbb{Z}_n like $\langle \bar{2} \rangle$ in \mathbb{Z}_{12} , and there are other cases where whole \mathbb{Z}_n is generated. This depends on the selection of the interval scale. In this paper, we place interval scale as a possible system to generate atonal music. From this viewpoint, the problem to be solved is which cyclic groups are generated depending on the selection of interval scales and under what condition whole \mathbb{Z}_n is generated. In this section, we investigate these problems. Solving these problems would enable us to know how to select interval scales.

First, we define a concept “chromatic” as a criterion of atonality:

Definition 5 (Chromatic). *The group generated by I_n is said to be chromatic if it is identical with whole \mathbb{Z}_n .*

The following lemma, which will be used later, is an extension of the famous “Euclid’s lemma.”

Lemma 1 (Generalized Euclid’s lemma). *There exists a combination of integers a_1, a_2, \dots, a_i such that $a_1y_1 + a_2y_2 + \dots + a_iy_i = d$, where y_k are integers ($i > 0, 1 \leq k \leq i$) and $\gcd(y_1, y_2, \dots, y_i) = d^5$.*

Proof. Mathematical induction is used to prove this statement. In the case where $i = 1$, either $a_1 = 1$ or $a_1 = -1$ is the solution because $\gcd(y_1) = |y_1|$. In the case where $i = 2$, the statement is true, because it is the ordinary version of Euclid’s lemma. From the prerequisite of the statement where $i = k + 1$, $\gcd(y_1, y_2, \dots, y_k, y_{k+1}) = d$. Let D be $\gcd(y_1, y_2, \dots, y_k)$, then $\gcd(y_1, y_2, \dots, y_k, y_{k+1}) = \gcd(D, y_{k+1}) = d$. From the statement where $k = 2$, we assume that there exists a combination of integers A, B such that $AD + By_{k+1} = d$. From the statement where $i = k$, we assume that there exist a combination of integers a_1, a_2, \dots, a_k such that $(a_1y_1 + a_2y_2 + \dots + a_ky_k) = D$. In $AD + By_{k+1} = d$, let’s substitute D , then $A(a_1y_1 + a_2y_2 + \dots + a_ky_k) + By_{k+1} = d$. We see that the coefficients of y_k are integers. Therefore, the statement is true in the case where $i = k + 1$. \square

Theorem 1 (Cyclic group $\langle \bar{d} \rangle$). *$\langle I_n \rangle$ is identical with the cyclic group $\langle \bar{d} \rangle$, where $I_n = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_i\}$, d is the greatest common divisor of $\{x_1, x_2, \dots, x_i\}$, and $0 \leq x_k \leq n - 1 (1 \leq k \leq i, i > 0)$.*

Proof. $\langle I_n \rangle \subseteq \langle \bar{d} \rangle$ is obvious, because any elements of I_n are the classes of multiple numbers of d . Conversely, if $\bar{d} \in \langle I_n \rangle$ is true, $\langle I_n \rangle \supseteq \langle \bar{d} \rangle$ is also true. Therefore, $\bar{d} \in \langle I_n \rangle$ is what we should verify here. From Lemma 1, there exists a combination of integers a_1, a_2, \dots, a_i such that $a_1x_1 + a_2x_2 + \dots + a_ix_i = d$. Therefore $a_1\bar{x}_1 + a_2\bar{x}_2 + \dots + a_i\bar{x}_i = \bar{d}$. Because the left-hand side is a member of $\langle I_n \rangle$, \bar{d} is also a member of $\langle I_n \rangle$. \square

In Lewin’s GIS (generalized interval system), intervals are defined on an abstract space, and the group of intervals on this space, which is denoted by IVLS, were studied [12, 13]. Interval system \mathbb{Z}_n is a specific case of IVLS. Morris made detailed studies about the groups that consist of tone-row transformations [14]. However, how groups are generated from multiple generators hasn’t been thoroughly studied.

⁵ For convenience, we define $\gcd(y_1) = |y_1|$ and $\gcd(0, 0, \dots, 0) = 0$.

This theorem means that $\langle I_n \rangle$ is completely determined and classified by d . For example, $\{\bar{0}, \bar{8}, \bar{10}\}$ and $\{\bar{6}, \bar{8}\}$ in \mathbb{Z}_{14} shares the greatest common divisor 2. Therefore, both interval scales generate the same subgroup $\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}, \bar{12}\}^6$.

Theorem 2 (Condition to be chromatic). *$\langle I_n \rangle$ is chromatic iff d and n are co-prime.*

Proof. $\langle I_n \rangle$ is chromatic. $\iff \langle \bar{d} \rangle = \mathbb{Z}_n$ (because of Theorem 1). $\iff \langle \bar{d} \rangle$ has $\bar{1}$. $\iff \exists B \in \mathbb{Z}$ s.t. $\bar{B}d = \bar{1}$. $\iff \exists A, B \in \mathbb{Z}$ s.t. $An + Bd = 1$. $\iff n$ and d are co-prime (partly because of Lemma 1). \square

This theorem can be used to judge whether I_n can generate atonal music that uses all of n pitch classes. In addition, if n is a prime number and $I_n \neq \{\bar{0}\}$, then $\langle I_n \rangle$ is always chromatic, because d and n are co-prime. This is highly suggestive since the nearest neighbors of 12, 11 and 13, are prime numbers. This is as if n avoids prime numbers. Contrary to prime numbers, 12 is the number whose number of divisors is the largest in natural numbers less than 24. Thanks to this fact, I_{12} can generate relatively large numbers of subgroups. This may have something to do with the reason why 12 has been the standard⁷.

If I_n contains two consecutive elements \bar{x} and $\overline{x+1}$, then $\langle I_n \rangle$ is chromatic because $d = 1$. This is the case of the interval scales of Ligeti and Berg. Here, we can guess the meaning of the selections of their interval scales. Although the smallness of the number of elements of an interval scale may contribute to clarify characteristics of sonority or sense of tonality, it may also prevent the group $\langle I_n \rangle$ from becoming chromatic because the smallness of the number of elements tends to increase d (see Theorem 1). However, the use of the consecutive interval scale ensures atonality. Therefore, we can interpret that the selections of Ligeti and Berg share a reasonable strategy to achieve a good balance between atonality and sense of tonality.

5. TONE ROW TRANSFORMATIONS AND INTERVAL SCALE

In this section, we investigate the relationship between tone row transformations and the interval scale⁸. Tone rows⁹ are usually related to their transformations such as transposition (T_i), prime (P), inversion (I), retrograde (R), and retrograde inversion ($RI = IR$). From the viewpoint of interval scale, one of the principal problems is how the interval scales of tone rows are transformed accompanied by the tone-row transformations. However, it is obvious that transpositions of tone rows don’t change the interval

⁶ The formula for calculating the order of $\langle I_n \rangle$ can be created, though we don’t deal with it here.

⁷ Based on this idea, selecting n that realizes local maximum values of a divisor function (a function that calculate the number of divisors of n) may be a good choice for microtonal music composition.

⁸ In general, interval scales are not necessarily used together with tone rows.

⁹ In this section, n is not necessarily twelve, and the number of notes in a tone row is not necessarily twelve nor n .

scales. Therefore, we only focus on a group of four transformations, which are P, I, R, RI , where P is the identity element of this group and does nothing to the tone row¹⁰.

First, let's formally define the interval scale of a tone row:

Definition 6 (Interval scale of tone row). *Let X_n be a tone row $X_n = [\overline{x_1}, \overline{x_2}, \dots, \overline{x_m}]$ ($\overline{x_k} \in \mathbb{Z}_n, 1 \leq k \leq m, m > 1$). The interval scale of a tone row X_n is denoted by $IS(X_n)$ and is defined as the set of differences of consecutive pitch classes, i.e. $\{\overline{x_k} - \overline{x_{k-1}} \mid 2 \leq k \leq m\}$.*

For example, $IS(X_6)$ of the tone row $X_6 = [\overline{0}, \overline{4}, \overline{3}, \overline{2}]$ is $\{\overline{4}, \overline{5}\}$. There is another key definition about a characteristic of interval scale:

Definition 7 (Symmetric¹¹). *An interval scale I_n is said to be symmetric, if I_n is invariant by the transformation of interval scale Inv , which replace each interval to the inverted interval (i.e. $I_n = Inv(I_n)$, $Inv(I_n) = \{0 - a \mid a \in I_n\}$).*

For example, $\{\overline{5}, \overline{6}, \overline{7}\}$ in \mathbb{Z}_{12} is symmetric because $Inv(\{\overline{5}, \overline{6}, \overline{7}\}) = \{\overline{5}, \overline{6}, \overline{7}\}$, and $\{\overline{2}, \overline{3}, \overline{4}\}$ in \mathbb{Z}_{12} is not symmetric because $Inv(\{\overline{2}, \overline{3}, \overline{4}\}) = \{\overline{8}, \overline{9}, \overline{10}\}$.

Theorem 3 (Condition of invariance).

(a) $IS(X_n) = IS(I(X_n)) \iff IS(X_n)$ is symmetric.

(b) $IS(X_n) = IS(R(X_n)) \iff IS(X_n)$ is symmetric.

Proof. (a): Let $Y_n = [y_1, y_2, \dots, y_{m-1}]$ be the series of interval between the consecutive elements of tone row $X_n = [x_1, x_2, \dots, x_m]$ ($m > 1, x_k \in \mathbb{Z}_n$). Similarly, the series of interval of $I(X_n)$ is $[-y_1, -y_2, \dots, -y_{m-1}]$. Therefore, $IS(X_n) = \{y_1, y_2, \dots, y_{m-1}\}$ and $IS(I(X_n)) = \{-y_1, -y_2, \dots, -y_{m-1}\}$. From these, the condition that $IS(X_n) = IS(I(X_n))$ means $\{y_1, y_2, \dots, y_{m-1}\} = \{-y_1, -y_2, \dots, -y_{m-1}\}$, i.e. $IS(X_n) = Inv(IS(X_n))$, which is indeed the definition of a symmetric interval scale.

(b): The series of interval between the consecutive elements of tone row $R(x) = [x_m, x_{m-1}, \dots, x_1]$ ($m > 1$) is $[-y_{m-1}, -y_{m-2}, \dots, -y_1]$ and $IS(R(x)) = \{-y_1, -y_2, \dots, -y_{m-1}\} = IS(I(x))$. In a similar way to the case of (a), therefore, the condition $IS(x) = IS(R(x))$ is equivalent to the condition that $IS(X_n)$ is symmetric. \square

From this theorem, we find that the use of the symmetric interval scale means that I and R can be used without "modulation" of the interval scale.

Theorem 4 (Retrograde inverse). $IS(x) = IS(R \circ I(x)) = IS(I \circ R(x))$.

Proof. Since the series of interval of $I \circ R(x)$ is $[y_{m-1}, y_{m-2}, \dots, y_1]$, $IS(I \circ R(x)) = \{y_{m-1}, y_{m-2}, \dots, y_1\} = IS(x)$. The second equality of the statement is derived from $I \circ R(X_n) = R \circ I(X_n)$. \square

This theorem shows that retrograde inverse always doesn't change the interval scale of the tone row.

¹⁰ These transformations form a Klein four-group whose operation is the composition of transformations.

¹¹ Rahn used the terms "inversionally symmetrical" [1]. However, it was used for pitch-class sets, not for sets of interval.

6. CONCLUSION

In this paper, we proposed the concept of "interval scale" and showed that it is a generating set of a subgroup of \mathbb{Z}_n . In addition, we showed that the group generated by an interval scale is characterized by d , the greatest common divisor of the interval scale, and also showed the necessary and sufficient condition for the group generated by an interval scale to be chromatic. From this condition, we found that the use of consecutive intervals may be a reasonable strategy to satisfy atonality and a sense of tonality, which are seemingly contradictory properties. Finally, we investigated the relationship between tone-row transformations and interval scale and showed the necessary and sufficient condition for the interval scale of a tone row to be unchanged by the group of transformations P, I, R , and IR . These results will provide a basic understanding and some criteria of selecting interval scales for composers who create music based on this concept.

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