N-gon Waves – Audio Applications of the Geometry of Regular Polygons in the Time Domain

Dominik Chapman
Goldsmiths, University of London
Department of Computing
Department of Music
mu102dc@gold.ac.uk

Dr Mick Grierson
Goldsmiths, University of London
Department of Computing
M.Grierson@gold.ac.uk

ABSTRACT

This paper presents an approach to the generation of periodic signals through the application of Euclidean geometry in the time domain. In particular, the content of this paper is focused on waveforms derived from regular polygons. It is an attempt to show how the geometrical relations and proportions of regular polygons and star polygons with respect to their Schläfli symbol can be made audible, and how these relations can be used in an acoustical or a musical context. A basic description is given of how to construct such geometrical waveforms and musical scales using the underlying geometry. In addition, this paper draws inspiration for its approach to synthesis and composition from experimental approaches to drawn graphical / ornamental sound. These include methods that came to prominence in Russia and Germany in the first part of the 20th century, such as those which used film and paper as primary media, and those that developed during the post-war period, including Oramics, and others. Most importantly, this paper describes a framework and examples that demonstrate processes whereby the geometry of regular polygons can be used to generate specific acoustic phenomena in frequency, timbre, phase, and metre.

1. INTRODUCTION

The relation of music and geometry can be argued to share a timeline alongside that of the creation of all musical instruments. Furthermore, geometry is used to describe acoustics, aspects of aesthetics, and is also used for theories and concepts in composition and perception of sound, for instance by Chladni[1], H. Jenny[2], D. Tymoczko [3], G. Mazzolla [4]. However, it seems that there is relatively little research which discusses the geometry of waveforms that are derived from, or related to, regular polygons. With the exception of the widely used sawtooth, triangle, and square wave, many texts on Fourier series usually do not cover other polygonal waveforms, although the theory to build and analyse them might exist.

During the first half of the 20th century, Russian and German composers[1] developed a range of techniques to generate sounds with ornamental patterns drawn on film. Two key publications which effectively detail the work of this period are *Sound in Z* by Andrey Smirnov [5] and *"Tones from out of Nowhere": Rudolph Pfenninger and the Archaeology of Synthetic Sound* by Thomas Y. Levin [6]. These both describe the work and experiments of a number of key Russian and German pioneers. Similar techniques were later used by Norman McLaren and Daphne Oram. Importantly, since the development of modern personal computers, graphical interfaces to control or draw sounds have become quite common in digital synthesisers[2].

This research began between 1999 and 2000 as an experiment into polymetric structures. Soon this work naturally led to the exploration of corresponding graphical representations. Further research resulted in the development of an approach to the visualisation of polymetric structures through the use of regular polygons in a circle as shown in Figure 1 on the left. These polymetric images are reminiscent of commonly used synthesised waveforms; sawtooth (figure 2), triangle (figure 3) and square (figure 3) waves[3]. This led to the study of whether more waveforms might be derived from regular polygons, and how they may relate to one another.

The first attempts to realize audio outputs from these waveforms as part of this project were produced on the Atari ST platform in Omikron Basic in 2000. Some time later, at Goldsmiths, University of London in 2012, the project was recoded in Processing, and then in JavaScript using the webAudio framework. The outcome was two prototype online synthesisers, with one still in an experimental stage. They both run only in the Google Chrome browser and are currently located at the following website:

http://igor.gold.ac.uk/~mu102dc/ngonwaves/start.html

There are also audio examples on the website which are useful for those interested in exploring the synthesis approaches detailed here.

[1] Russian composer Arseny Avraamov presented his first graphical sound experiments in 1930. German composers that worked with drawn sounds around the same time were Rudolf Pfenninger and Oskar Fischinger.
[2] For instance, the wavetable synthesisers by PPG and Waldorf or the Fairlight synthesisers.
[3] N-gon waves in all the figures, except figure 10 and 17, of this paper are shown as bold black connected lines.

The two synthesiser prototypes are called the “N-gon Wave Synthesizer” and “N-gon Wave Synthesizer Sequencer”. Polygons are often referred to as “n-gons”, hence we chose the term “n-gon waves” for the polygonal waveforms.

Fundamentally, a unit circle or unit frequency is used to derive the waveforms, from which a variety of scales and timbres can be generated. To some degree this technique enables the representation of geometrical relations in sound. Some of the results of these experiments are presented here.

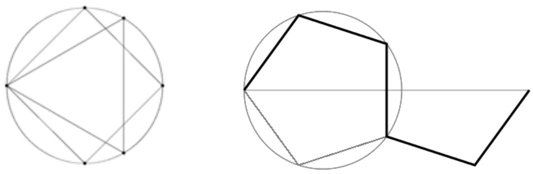

Figure 1. Polymetric structures visualized as polygons in a unit circle (left). A pentagon wave with start phase 0 is derived from a pentagon (right).

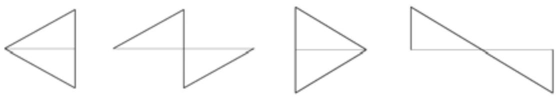

Figure 2. Sawtooth waves or trigon waves with start phase 0 and π/n derived from a triangle.

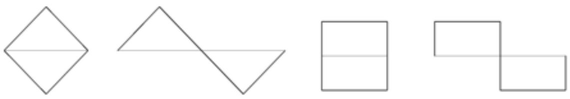

Figure 3. A triangle and a square wave or tetragon waves with start phase 0 and π/n derived from a square

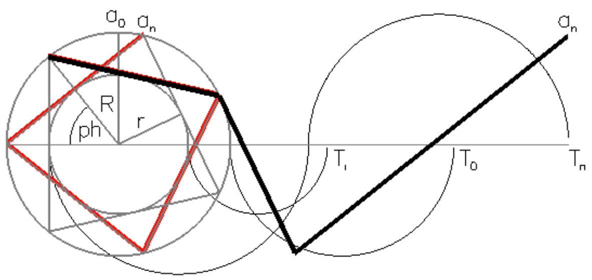

Figure 4. A heptagram wave (black) with start phase 2π/n and four edges is derived from four edges (red) of a heptagram (grey) with Schläfli symbol {7/2}. The figure shows variables (a_0 , a_n , φ = ph , R , r , T_0 , T_i , T_n) that are used for n-gon wave calculations explained in sections 2 and 3 of this paper. The grey circle waves are derived from the circumcircle (wavelength = T_0), the incircle (wavelength = T_i), and the wavelength (or frequency) of the heptagram wave ($\varphi = \pi$, wavelength = T_n).

2. BASIC DESCRIPTION OF A N-GON WAVE

The following basic description employs formulas that can be used directly to generate or program n-gon waves. It is not intended as a complete example, instead it focuses on the central approaches used for the construction of n-gon waves.

It has to be pointed out that the waveforms described here are not polygon sine waves. Although a similar approach, the waves presented here are angular and are not curved and smooth as polygon sine waves.

One way to construct a n-gon wave is to cut a regular polygon in two halves along a line from a vertice to the middle of its opposite edge, and then rotate one half either around the vertice or the edge, as shown in figures 2 and 3. This method works fine with polygons with Schläfli-Symbol {n} as can also be seen in figure 1 on the right. Another approach needs to be used to generate waveforms derived from star polygons with fractional Schläfli-Symbols {n/q}, as depicted in figure 4. In this case a regular polygon or a star polygon is unwrapped or unfolded. The following description focuses on this second approach.

2.1 Variables and Basic Calculations

The following variables (see also figure 4) are used for the construction of a n-gon wave:

S = Sample rate
n = The number of vertices or edges of a regular polygon, Schläfli symbol {n} (the letter p is also used instead of n for the Schläfli symbol of star polygons {p/q})
q = The stellation of a regular polygon, i.e. the edge or line that connects every qth vertice of the n vertices on the circle periodically, it represents which and how vertices are connected, for in stance every second vertice or every third vertice, etc., it is the denominator In the Schläfli symbol {n/q} (or if p instead of n is used {p/q})
l = The number of connected edges or lines
φ = Start phase
T_0 = Wavelength of the unit circle wave in samples
T_i = Wavelength of the incircle wave in samples
T_n = Wavelength of the n-gon wave in samples
f_0 = Fundamental frequency of the unit circle wave
f_n = Frequency of the n-gon wave
a_0 = Amplitude of the unit circle wave
a_n = Amplitude peak of the n-gon wave
Ω = Unit-Circle Exponent
ε = Frequency-Ratio Exponent
R = Radius circumcircle
r = Radius incircle
r_n = Radius of the n-gon wave unit circle with frequency f_n
m = Gradient

The calculations for the radius of the circle and the incircle are:

$$R = S / 4f_0 \quad (1)$$

$$r = R\cos(q\pi/n) \quad (2)$$

The following formula is used to calculate the gradients for the interpolation between the vertices of the n-gon wave:

$$m(k) = \sum_{k=0}^{n-1} \frac{a_0 [\sin(2\pi q(k+1)/n + \varphi) - \sin(2\pi qk/n + \varphi)]}{|R[\cos(2\pi qk/N + \varphi) - \cos(2\pi q(k+1)/n + \varphi)]|} \quad (3)$$

If n gets larger, the n-gon wave converges to a polygonal circle wave. A polygonal circle wave is not the same as a sine wave as can be seen from figure 5. A polygonal circle wave is comprised of partial sine waves and has a distinct sound. The frequency spectrum of a polygonal circle wave is illustrated in figure 6. In this paper, the "polygonal" is discarded and the term "circle wave" is used to refer to a polygonal circle wave.

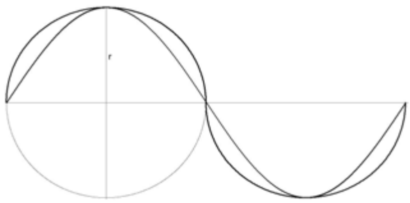

Figure 5. A polygonal circle wave and a sine wave of the same frequency.

3. FREQUENCY, AMPLITUDE, PHASE, AND TIMBRE - THE UNIT CIRCLE AND THE UNIT FREQUENCY

3.1 Frequency and Unit Circle

If a n-gon wave is derived from a unit circle (that can be the circumcircle or the incircle) and the two basic start phases ($\varphi = 0$, $\varphi = \pi/n$) are used, there are three cases to be considered for the calculation of its frequency: if n is even and the start phase $\varphi = 0$, if n is even and $\varphi = \pi/n$, and if n is odd and the start phase is $\varphi = 0$ or $\varphi = \pi/n$. The frequencies can be calculated with these formulas:

First calculate T_n

$$T_n = \begin{cases} 4R, & n = \text{even and } \varphi = 0 \\ 4r, & n = \text{even and } \varphi = \pi/n \\ 2R + 2r, & n = \text{odd and } \varphi = 0 \text{ or } \varphi = \pi/n \end{cases} \quad (4)$$

Or alternatively, especially if Schläfli symbols of the form {n/q} are used for star polygons:

$$T_n = \sum_{k=0}^{n} |R[\cos(2\pi qk/n + \varphi) - \cos(2\pi q(k+1)/n + \varphi)]| \quad (5)$$

Then calculate:

$$f_n = S / T_n \quad (6)$$

A n-gon wave can also be transposed to any frequency f_n with respect to recursion exponents (Ω, ε) and recalculations of the radius r_n for the transposed wave, like this:

$$r_n = \frac{R[1/\cos(q\pi/n)]^{\Omega}}{(T_0/T_n)^{\varepsilon}} \quad (7)$$

Then replace R with r_n in calculation (4) or (5) to recalculate T_n. For instance, to calculate T_i set ε to 1 and multiply r_n (the new R) by 4.

3.2 Amplitude

The peak amplitudes of the n-gon waves are also dependent on the start phases and the number of vertices, or edges, respectively. As the amplitude of the unit circle wave a_0 is set to its radius R, the amplitude peaks of the related n-gon waves decrease for higher frequencies or increase for lower frequencies proportionally to R. This has an impact on the amplitude peaks produced by the waveforms, which affect their timbre. The focus of this paper is on the frequencies (scales) and the timbre in isolation from the structure of the amplitude variations, which will be detailed in future publications.

3.3 Phase

So far, two basic start phases are often used, as described above: $\varphi = 0$ and $\varphi = \pi/n$. The difference that these two phases make can be seen in the image of the tetragon waves in figure 3. A start phase $\varphi = 0$ generates a triangle wave, whereas a start phase of $\varphi = \pi/n$ produces a square wave. Both waves have different frequencies if they are derived from the same unit circle and same frequency-ratio exponents.

The phase can also be used to generate polymetric phase rotation. If the phase of different n-gon waves changes with the same speed, polymetric patterns can be perceived. This corresponds to the image (figure 1) that led to this experiment with n-gon waves.

With an almost symmetric n-gon wave, such as a star polygon wave, and the matching n, q, l and φ settings, it is possible to generate an audio effect similar to a n-gon wave being played forward or backward.

3.4 Timbre and Unit Frequency

If different n-gon waves have the same unit frequency, the differences in the timbre or the harmonic spectrum of the waveforms can sometimes be heard and compared quite well. The timbres of n-gon waves range from the common triangle, sawtooth and square wave like sounds for n-gon waves with Schläfli-Symbol {n} to coarser and percussive sounds with Schläfli-Symbol {n/q}. As mentioned before in 3.3 the start phase contributes also to the timbre of a n-gon wave.

Through a basic analysis of n-gon waves with a Discrete Fourier Transform (DFT), followed by resynthesis with an Inverse Discrete Fourier Transform (IDFT), it appears that in the frequency domain, if n is odd all partials seem to be used for building the wave, and when n is even odd numbered partials seem to be much more present. Figure 6 shows the amplitude, phase spectrum of a circle wave, and the resynthesized circle wave.

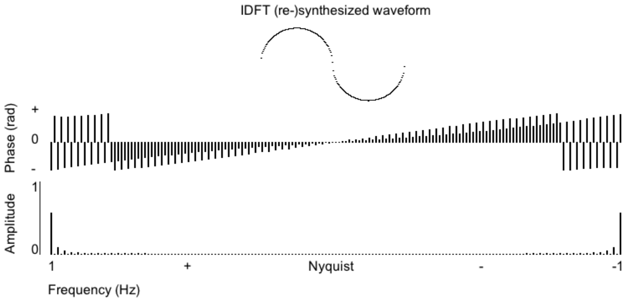

Figure 6. The phase and amplitude spectrum of a circle wave, and above the resynthesized circle wave.

3.5 N-gon Wave Drum Sound Example

Here is a short example of how to set the parameters to synthesize a drum like sound with a n-gon wave derived from a star polygon with Schläfli symbol {420/209}:

$n = 420$ (a large n for star polygons)
$q = n/2 – 1 = 209$ (also n/2, more melodic)
$l = n/4 = 105$ (or higher or lower, affects the timbre)
$\varphi = \pi/2 = 2\pi/n(n/4)$ (changes affect the timbre)
$f_0 = 55$Hz (higher or lower values for different pitches)

Figures 7 and 8 show parts of the same n-gon wave. In figure 7 the circle seems to be filled with the grey color that is used for the edges of the polygon in the circle. This is due to the large number of 420 vertices or edges that are used. The actual n-gon wave is much longer than seen here.

Figure 8 shows a magnified version of a smaller part of the same n-gon wave where the edges of the star polygon can be identified. As can be seen in figure 7, if a star polygon with a large n (i.e. a large number of vertices or edges) is used to generate a n-gon wave similar to this one, the gradient decreases for each edge. At some limit, i.e. if n would be infinite, it might converge to 0.

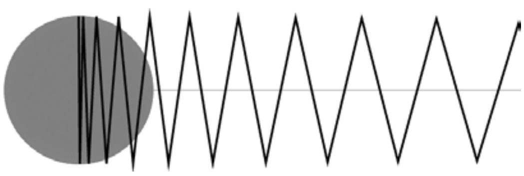

Figure 7. A n-gon wave with Schläfli symbol {420/209}, start phase $\varphi = \pi/2$, and number of connected edges $l = 105$. The wave has a drum like timbre. Because a star polygon with 420 vertices or edges is used, the circle seems to be a grey dot.

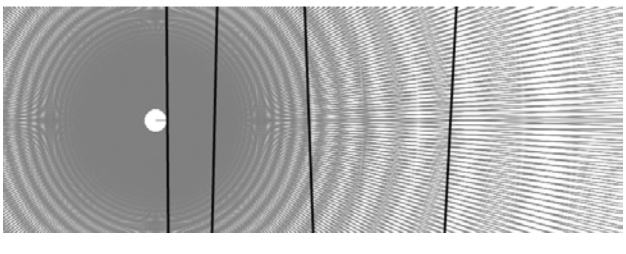

Figure 8. A magnified image of the n-gon wave shown in figure 8. Parts of the edges of the star polygon in the unit circle are visible.

4. SCALES

As should appear obvious, every n-gon wave can oscillate with every frequency and they can be used in every kind of scale, albeit some scales seem to be inherent to the subject itself.

The scales presented here should be seen as an approach to make the geometrical relations and proportions of regular polygons and star polygons audible. The n-gon wave scales described here are derived from regular polygons and star polygons and a unit circle (i.e. all n-gon waves are derived from the same circumcircle radius) or a unit frequency ratio (i.e. the ratio of the circle wave wavelength and the n-gon wavelength) of the corresponding circle wave.

Other properties of regular polygons could be chosen as unit to derive n-gon scales from, for example, a unit incircle, a unit edge or a unit stellation line that connects the vertices of a star polygon. It is possible to build scales from the phases of the stellations of a star polygon if they are used as start phases (φ).

More than one unit parameter or other geometric properties of regular polygons or star polygons can also be used for the construction of n-gon wave scales. For instance, unit circle recursion, unit frequency ratio recursion, and the stellations of a star polygon can be combined into a scale.

4.1 Unit Circle Scales

Unit Circle Scales are comprised of n-gon waves with $\varphi = 0$ and $\varphi = \pi/n$ derived from n-gons adjacent to one or more unit circles. A unit circle is used as a centre frequency of a corresponding circlewave of which the other n-gon wave frequencies of the scale are derived from. The unit circle wave figures as a form of axis to which the n-gon waves of a Unit Circle Scale are adjacent to.

Here is a first example. The range of one octave and one additional higher semitone can be constructed with two trigon waves and two tetragon waves. To derive the fundamental frequency of the scale, a trigon with start phase 0 on the outside adjacent to the unit circle is used (unit circle exponent $\Omega = 1$). For its octave a trigon with startphse π/n on the inside adjacent to the unit circle is used (unit circle exponent $a = 0$). The ratio of the circle wave and the trigon waves will then be the fifth on the outside, and fourth on the inside, with the ratios 3/4 for the first and 3/2 for the octave trigon wave. The two tetragon waves are used to build another octave around the circlewave which then becomes their tritone, the ratios of the tetragon waves and the circle wave are: $\sqrt{2}/2$ and $\sqrt{2}$, respectively.

All n-gon waves with an odd n have different frequencies. The two start phases $\varphi = 0$ and $\varphi = \pi/n$ do not change the frequency or the amplitude peaks of the n-gon waves with an odd n, but the start phases do change the frequencies and the amplitudes of the even n n-gon waves.

All n-gon waves with an even n on the inside of a unit circle with start phase $\varphi = 0$ and unit circle exponent $\Omega = 0$ have the same frequency as the unit circle wave. If the unit circle exponent is changed to 1, so that the polygons lie on the outside adjacent to the unit circle, they have different frequencies.

All n-gon waves with an even n on the inside of a unit circle with start phase φ = π/n and unit circle exponent a = 0 have different frequencies, too.

All of the n-gon waves with the same n inside and outside adjacent to the unit circle of the circlewave seem to show reciprocal frequencies; for instance, trigon waves with 1/(3/2) = 3/4 and tetragon waves with 1/√2 = √2/2, etc.[4]

A scale of ascending or descending frequencies comprised of n-gon waves can be built with the two start phases and the polygons on the inside and outside adjacent to the unit circle as described above. The above mentioned semitone that is higher than the octave of the trigon wave is the octave of the tetragon wave with start phase φ = 0 and unit circle exponent Ω = 1. It is a tetragon wave with start phase φ = π/n and unit circle exponent Ω = 0. The scale does not ascend or descend with one frequency per n of a n-gon wave, it rather jumps back and forth between odd and even n frequencies. For instance, the tetragon wave is always higher than the trigon wave, whether it is on the inside or on the outside of the unit circle. With the exception of the tri- and tetragon wave, a wave with odd n and a unit circle exponent a = 1 on the outside of the unit circle is followed by a lower frequency of the next higher even n. With n-gon waves with odd n and a unit circle exponent a = 0 on the inside of the unit circle it is the other way round: the frequency of a wave with even n will be followed by a lower frequency of a wave with the next higher odd n. If the n values of the n-gon waves are arranged symmetrically around the circle wave, but the start phase and the unit circle exponent change as described above, the frequency ratios seem to be reciprocally mirrored around the unit circle of the circle wave.

The larger the numbers that are used for n, whether they might be even or odd, the closer the n-gon wave gets to the circle wave. It seems that even and odd n will converge into a unit circle when they reach infinity. This raises some questions about whether the n of the unit circle wave is even or odd when n reaches infinity (n = ∞), whether there are two inifinities (one for even and one for odd) or if there are other maybe paradox solutions. Unfortunately, we are not able to give an answer to these questions here.

Scales that use only odd or even numbers for n might jump less forwards and backwards than a scale that uses all or a set of even and odd numbers.

Table 1 shows the frequencies of a unit circle scale of 13 frequencies of one octave plus the one extra frequency of the tetragon wave, the start phases, the unit circle exponents, the frequencies of a twelve tone equal temperament scale, and the frequencies of the just intonation scale. All three scales start from 1Hz.

A Unit Circle Scale can also contain more than one unit circle. The unit circles and their circle waves can be arranged geometrically derived from polygons or n-gon waves. For example, octaves and fourths can be derived from trigon waves and used as new unit circle centres frequencies.

n	φ	Ω	$f_{n\text{-gon}}$	$f_{equal\ temp}$	just
3	0	1	1.0000	1.0000	1.0000
4	0	1	1.0607	1.0594	1.0666
5	0	1	1.3416	1.1224	1.1250
6	0	1	1.2990	1.1892	1.2000
7	0	1	1.4218	1.2599	1.2500
8	0	1	1.3858	1.3348	1.3333
∞, unit	0 or π/n	0 or 1	1.5000	1.4142	1.4000
8	π/n	0	1.6236	1.4983	1.5000
7	π/n	0	1.5781	1.5874	1.6000
6	π/n	0	1.7320	1.6817	1.666
5	π/n	0	1.6583	1.7817	1.777
4	π/n	0	2.1213	1.8877	1.8750
3	π/n	0	2.0000	2.0000	2.0000

Table 1. A Unit Circle Scale derived from one circle wave. n = number of vertices or edges of the polygon, φ = start phase, a = unit circle exponent, $f_{n\text{-gon}}$ = frequency n-gon wave, $f_{equal\ temp}$ = frequency equal temperament scale, just = frequency just intonation scale. All frequencies are in Hertz.

n	φ	Ω	$f_{n\text{-gon}}$	$f_{equal\ temp}$	just
∞, unit	0 or π/n	0 or 1	1.0000	1.0000	1.0000
8	π/n	π/n	1.0824	1.0594	1.0666
7	π/n	π/n	1.0520	1.1224	1.1250
6	π/n	π/n	1.1547	1.1892	1.2000
5	π/n	0	1.1056	1.2599	1.2500
4	π/n	0	1.4142	1.3348	1.3333
3	0 or π/n	0 or 1	1.3333	1.4142	1.4000
4	0	1	1.4142	1.4983	1.5000
5	0	1	1.7889	1.5874	1.6000
6	0	1	1.7320	1.6817	1.666
7	0	1	1.8958	1.7817	1.777
8	0	1	1.8477	1.8877	1.8750
∞, unit	0 or π/n	0 or 1	2.0000	2.0000	2.0000

Table 2. A Unit Circle Scale derived from two circle waves. n = number of vertices or edges of the polygon, φ = start phase, a = unit circle exponent, $f_{n\text{-gon}}$ = frequency n-gon wave, $f_{equal\ temp}$ = frequency equal temperament scale, just = frequency just intonation scale. All frequencies are in Hertz.

[4] Although n-gon waves with an odd n seem to show almost reciprocal values, irregularities can be observed. These irregularities could be caused by rounding errors by the computer used for the calculations of the frequencies. As it it is not clear where they come from this will need some further investigation.

A Unit Circle Scale can also contain more than one unit circle. The unit circles and their circle waves can be arranged geometrically derived from polygons or n-gon waves. For example, octaves and fourths can be derived from trigon waves and used as new unit circle centres frequencies.

Table 2 shows the frequencies (starting from 1Hz) of a unit circle scale of 13 frequencies derived from two unit circles. One unit circle is halve the size of the other, i.e. two circle waves with a frequency ratio of one octave. The parameters and scales in this table are the same as in table 1, but as can be seen from the table, the values for n, φ, Ω, and $f_{n\text{-}gon}$ are different. The frequency ratio comparison to the other scales (equal temperament, just intonation) shows other parts of the frequency properties of a Unit Circle Scale than the one in table 1.

4.2 Unit Circle Recursion Scales

The following recursive process is used to generate Unit Circle Recursion Scales:

If the incircle of a polygon is used as a new unit circle for another polygon inside this circle, and this process will be repeated for a number of times, a musical scale can be generated with the derived n-gon waves of each circle. The same method can also be inverted so that the unit circle or circumcircle of one polygon becomes the incircle of another polygon, etc.

Mathematically this means that the Unit Circle Exponent Ω in equation (7) is increased or decreased for each new frequency of the musical scale. Decreasing Ω increases the frequency and increasing Ω decreases the frequency.

$$\Omega_{n+1} = \Omega_n \pm 1 \quad (8)$$

If a trigon wave is used for each step of the process, a scale of octaves is generated. If a tetragon wave is used, a scale of tritones is produced. It can be concluded that the larger n is (the number of vertices or edges) the smaller are the steps of the recursion scale. Figure 9 illustrates an Unit Circle Recursion Scale with trigons.

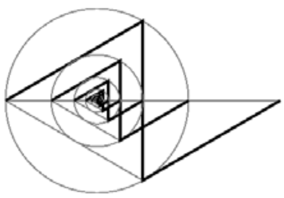

Figure 9. A trigon Unit Circle Recursion Scale of octaves.

4.3 Unit Frequency Ratio Recursion Scales

The same recursive process as for Unit Circle Recursion Scales can be applied to the frequency ratio of a unit frequency. The frequency of a n-gon wave becomes the new unit frequency for the next n-gon wave frequency calculation, etc. It can also be described as powers of the ratio of the circle wave wavelength to the n-gon wavelength. This corresponds to an increment or decrement of the Frequency-Ratio Exponent ε in equation (7). Decreasing ε also decreases the frequency and increasing Ω also increases the frequency.

$$\varepsilon_{n+1} = \varepsilon_n \pm 1 \quad (9)$$

If again a trigon wave is used for each step of the process, a circle of fourths is generated due to the frequency ratio of 4:3 of a trigon wave and a unit circle frequency.

For even numbered n-gon waves the start phase has to be set to φ = π/n. Otherwise, the recursion frequency ratio will always be 1.

If a tetragon wave is used and its start phase is set to φ = π/n, a scale of tritones is produced. Note that this is the same scale as the Unit Circle Recursion Scale of tetragon waves.

Again, as with the Unit Circle Recursion Scale, the larger n is (the number of vertices or edges) the smaller are the steps of the recursion scale. Figure 10 shows the Trigon Unit Frequency Ratio Recursion Scale of a circle of fourths and the Unit Frequency Ratio Recursion Scale of an octagon wave

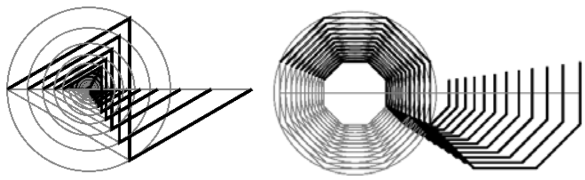

Figure 10. A trigon Unit Frequency Ratio Recursion Scale of a circle of fourths (left) and an octagon Unit Frequency Ratio Recursion Scale (right).

4.4 Edge Scales

Edge Scales can be constructed if the number of edges and vertices of the regular polygon stay the same, but the number of edges of the derived n-gon wave is increased or decreased.

In this paper, the variable l is used for the number of edges because this is the first letter in the word “line”. Actually, the line is the edge of a polygon, hence the Edge Scale could also be called the L Scale.

Edge scales show a periodic behaviour. If l is a multiple of n, the frequency is the same. If it is not a multiple of n and if l is increased, the frequency decreases.

The start phase (φ) makes a difference for even numbered n-gons. If φ = π/n, two steps of the scale are the same because of the symmetry of the vertical edges. If φ = 0, all the steps of the scale are different.

If a unit frequency is used, Edge Scales seem to change the partial frequencies of a n-gon wave spectrum.

Figure 11 illustrates an Edge Scale of a hexagon with φ = 0.

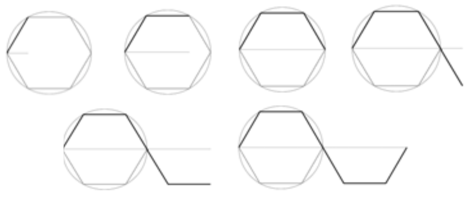

Figure 11. A hexagon Edge Scale or L Scale with φ = 0.

4.5 Stellation Scales

Stellation Scales are derived from a regular polygon and its stellations.

If different q values in the Schläfli symbol{n/q} of a regular polygon are used to build its stellations, a corresponding scale can be generated.

For instance, if n is 12 and φ = 0, the scale is comprised of a dodecagon wave, 5 star polygon stellation waves, and their two basic start phases 0 and φ/n. That makes a total of 12 different n-gon waves with only the half of them with an individual frequency. Figure 12 shows 6 n-gon waves as dodecagon wave stellations of the scale. Note that three of these n-gon waves are derived from regular polygons of star figures, only one is derived from a dodecagram, and one n-gon wave is a straight horizontal line that is not depicted in full length.

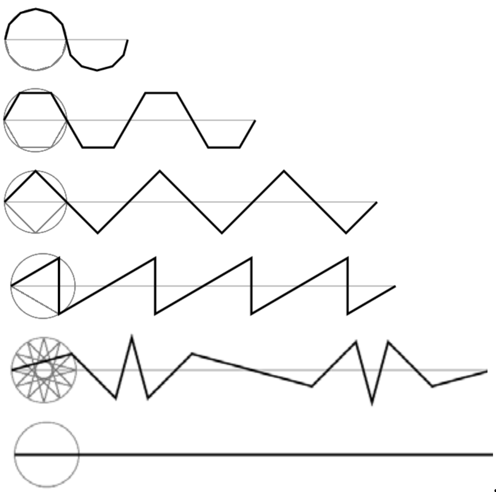

Figure 12. This figure shows six n-gon waves of a dodecagon Stellation Scale. Three of these n-gon waves are derived from regular polygons of star figures, only one is derived from a dodecagram. Note that one n-gon wave is a horizontal line and it is not depicted in full length.

4.7 Phase Rotation Scales

If the start phase (φ) of a regular polygon or a star polygon is increased or decreased by 2π/n, so that the start phases become multiples of 2π/n as shown in equation (10), scales can be constructed with start phases that seem to rotate around the incircle corresponding to the vertices of the polygon.

$$\varphi(k) = \sum_{k=0}^{n} k(2\pi/n) \qquad (10)$$

If a regular polygon is used, the value of l (lines that connect the vertices of the polygon) is equal to n (number of vertices of the polygon), and q (the stellation of the star polygon) is 1, a phasing effect occurs. If l is not equal to n, a scale of different frequencies can be constructed.

From a star polygon phase rotation scales can be constructed in the same way as with regular polygons.

Other incremental values for φ that do not match the vertices of a polygon can be chosen, too, for example logarithmic values. Figure 13 shows four of twelve n-gon waves of a Dodecagram Phase Rotation Scale with n = 12, q = 5, and l = 6.

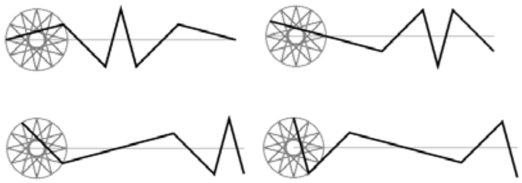

Figure 13. Four of twelve n-gon waves of a dodecagram Phase Rotation Scale with n = 12, q = 5, and l = 6.

4.8 Intervals and Chords

It is possible to build intervals or chords from the above described scales. If n-gon waves are derived from the same unit circle, some intervals seem to be suitable to be used in a harmonic context. Because this needs some further investigation, we can only give a few examples at the moment. For instance, trigon and pentagon, trigon and heptagon waves, intervals of the nonagon Unit Circle Recursion Scale or the pentagon Unit Frequency Ratio Recursion Scale can all be used to build chords.

Some of the intervals or chords that can be built from the frequencies of a Unit Circle Scale seem to resemble the ones from the equal temperament or just intonation scale.

For instance, the interval of a tetragon wave and hexagon wave (ratio = 1.2246) is close to a major third interval of the equal temperament scale (ratio = 1.2599). The ratio of the tetragon wave and the circle wave of a Unit Circle Scale is equal to the tritone of the equal temperament scale: $\sqrt{2}$ = 1.1412.

With a ratio of 1.3333 the interval of the circle wave and the adjacent trigon wave inside the circle is the same as a perfect fourth, or with a ratio of 1.5 for an adjacent trigon wave outside the circle a perfect fifth of the just intonation scale.

5 EVALUATION

5.1 Aliasing, Low Frequencies, Computation Time, Software Applications, and Recordings

In this research, angular n-gon waves have mostly been used, chiefly due to a desire to test both their acoustic properties and their graphic representation. Although the angularity of the waveforms introduces aliasing, it is the angularity that contributes to the computation of geometric proportions that can be used in an acoustical or musical context; for instance, scales derived from geometric proportions of regular polygons as described in section 4 of this paper. N-gon waves can be used as waveforms for additive synthesis, subtractive synthesis, and modulation (experiments with additive n-gon wave synthesis and modulation were conducted by the authors but are not documented in this paper).

The aliasing is introduced by the sharp edges of the computer generated waveforms. The high frequency partials of a n-gon wave frequency spectrum that are used to generate sharp edges can be higher than the Nyquist frequency. To avoid aliasing these frequencies could be filtered out with a low-pass filter.

Sometimes very low frequencies might occur while working with n-gon waves. A high-pass filter can be used to filter them out or they may be used as modulation signals.

An implementation of a n-gon wave oscillator algorithm (programming language C++) was run on a laptop to test and estimate the computational time (i.e. the time complexity) of the algorithm. It appeared that the computation time[5] of a star polygon wave (the drum sound example in section 3.5) with Schläfli symbol {420/209}, $\varphi = \pi/2$, and with a frequency of 0.25Hz was on average about 0.0029 seconds. In the worst case it was 0.0063 seconds, which is about 160.875Hz, or 9’652.51bpm (beats per minute). This indicates that the computation of this low frequency n-gon wave was about 643.5 times faster than the frequency of the wave itself. The computation time (about 0.0003 seconds or 3’333.3333Hz) of the same waveform with a higher frequency of 4048Hz turned out to be slower on average than the frequency of the wave itself. The computation time average of 1000 test waves with octave frequencies (2^x Hz) from 1Hz to 16’384Hz and variable settings for n and q was slightly higher than approximately 0.0003 seconds (about 2’989.3132Hz or about 179’358.7923bpm). This computation time average appeared to figure as a performance limit of the n-gon wave algorithm that was run on the test computer. The limitation needs to be considered if the n-gon wave algorithm is implemented in software applications. The implementation of the tested algorithm could probably also be optimised, for instance, with inline assembly code. However, as this computation time limit is higher than most tempos used in common sequencer software, it seems that the algorithm can be implemented in such software applications. [6]

To test the computational accuracy and potential data loss of the output of prototype n-gon wave synthesisers recordings were made with common recording software. Figure 14 shows a prototype n-gon wave synthesiser output recording of a tetragon wave with $\varphi = \pi/4$, and hexagon wave with $\varphi = \pi/6$, i.e. both with $\varphi = \pi/n$. The recording was made with Audacity[7]. It appeared that the recordings of an online prototype n-gon wave synthesiser were occasionally distorted or interrupted when the recordings were made on the same computer as the online synthesiser software was run on. It is not clear whether these issues were due to the implementation, the speed of the internet connection, or due to the speed of computation available on the test computer. The online synthesiser performed well if no recording software was used on the same computer at the same time. With the offline versions of the prototype n-gon wave synthesisers there appeared to be no such issues so far. It was possible to run offline n-gon wave synthesiser software and common recording software on the same computer (laptop) while algorithmically and/or manually changing the synthesiser parameters at the same time.

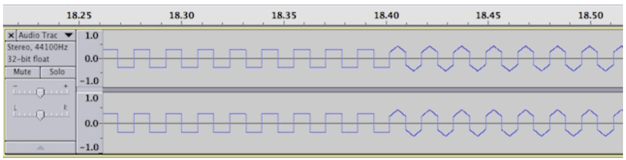

Figure 14. A recording with Audacity of tetragon and hexagon waves with $\varphi = \pi/n$ generated with a protoytpe n-gon wave synthesiser.

6 CONCLUSIONS

As documented in this paper, n-gon waves can be used as a means to approach the expression of geometrical proportions and relations of regular polygons and star polygons in sound. Although aliasing might occur some geometric relations are still audible, and the waveforms may be used for additive synthesis, subtractive synthesis, and modulation. N-gon waves offer a variety of musical scales, harmonic relations, timbres, and other properties to experiment with. Frequency, amplitude, and phase can be used as independent variables or can additionally depend on one another through the geometrical properties of regular polygons. N-gon waves can be played in an equal temperament scale. However, with their inherent geometrical properties they can also be used as a means to explore other possibilities in sound.

Acknowledgments

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6 REFERENCES

[1] E.F.F. Chladni, *Die Akustik*, Olms Verlag, 1802, Reprint: Hildesheim, 2004.

[2] H. Jenny, *Kymatik, Cymatics*, Basilius Presse, 1967.

[3] D. Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. Oxford University Press, 2011.

[4] G. Mazzolla, *Geometrie der Töne: Elemente der mathematischen Musiktheorie*, Birkhäuser, 1990.

[5] A. Smirnov, *Sound in Z*, Sound and Music London, 2013.

[6] T.Y. Levin, “*Tones from out of Nowhere”: Rudolph Pfenninger and the Archaeology of Synthetic Sound*, Grey Room 12, Summer 2003, pp. 32–79. Grey Room, Inc. and Massachusetts Institute of Technology, 2003.

[5] The results are rounded to four decimal places.

[6] For instance, the online “N-gon Wave Synthesizer Sequencer” mentioned in the introduction of this paper combines a n-gon wave oscillator and a sequencer. (A Max and/or Pure Data external is currently in development by the authors).

[7] The Audacity recording freeware can be found here: http://audacity.sourceforge.net/