

The “Harmonic Walk”: an Interactive Educational Environment to Discover Musical Chords.

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ABSTRACT

Harmony has always been considered a difficult matter to learn, also by experienced musicians. The aim of this paper is to present a system designed to provide unskilled users with an indication about the sound of the different harmonic regions and to help them to build a cognitive map of their relationships, linking musical perception to spatial abilities like orientation and wayfinding. The *Harmonic Walk* is an interactive environment which responds to the user’s position inside a rectangular space. Different chords are proposed to the user depending on her/his position. The user’s task is to find and to recognize them, and, then, to decide how to link the chords producing a convincing harmonic progression. This can be made by choosing a precise path to perform the best satisfying “harmonic walk”, selecting it among various possibilities. From a theoretical point of view the project is inspired to the neo-Riemannian ideas of harmony and parsimonious progressions, which try to give a wider and coherent framework to 19th century harmony and to its representation. The results of our preliminary tests confirm that, in a sample of children from 7 to 11 years old, most of the participants were able to locate the chords and to find some valid path to perform a harmonic progression.

1. INTRODUCTION AND RELATED WORK

Harmony has always been considered a difficult matter to learn, also by experienced musicians. One of the main reasons of this difficulty is that everybody can listen to a harmony or can perceive a change between two chords, but few people are able to play a simple harmonic progression even among professionals. To do so, one must be able to play a polyphonic instrument and to read music, or, alternatively, must have some good keyboard improvisation skills. Indeed, some people can play harmonies by heart or by ear, but very few have some consciousness of what they are playing. Consequently, if we exclude guitar and accordion players, people in general have no physical feedback of what playing harmony means. This greatly contributes to make musical harmony a matter very far away from a common user’s every day experience.

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Figure 1. The *Harmonic Walk* while being tested by an elementary school boy.

The harmonic relationships have been traditionally represented using spatial schemas like the tonnetz (see Subsec. 2.1). Moreover, many harmonic features are expressed by means of spatial metaphors. Concepts like “harmonic progression” imply the idea of a series of consecutive steps which start from a point and which follow a concatenation sequence towards an end; the “harmonic turnaround” is another popular term to indicate the metaphor of a chord pattern cyclical repetition, to remember only the commonest ones. Such kind of features are usually learned and understood by professional musicians in years of study and practice. Our idea is that transferring such metaphors on a physical space is a way of making these abstract concepts concrete, providing the user with the actual embodied cognition of what harmonic structures are.

Summarizing, the aims of the *Harmonic Walk* project are:

1. to put unskilled users in condition of entering the surface of a physical tonnetz, providing them a spatial and aural feedback of what the harmonic relationships are, in an easy and handy way (see Fig. 1);
2. to study the cognitive processes by which users can build an inner map of the sound qualities of the chords linked by the tonnetz spatial relationships;
3. to consider the possibility of extending the spatial reasoning implied in the *Harmonic Walk* to other fields that employ spatially tagged objects, for instance the geographic learning, or the study of the geometrical forms.

In this perspective and at this early stage of the project, we want to investigate some preliminary aspects of the application. In particular this paper aims at discovering if an unexperienced user is able to recognize spatial locations by means of a harmonic feedback, to remember and to link them with one or more paths. Secondly we are interested in observing if and how world's and body's asymmetries influence the choice of the linking paths. After a short review of similar projects reported in literature, we provide a theoretical background about harmonic space theories (Subsec. 2.1) and about the interaction with the physical space (Subsec. 2.2). In Sec. 3 we describe the application architecture and in Sec. 4 we present some tests results.

1.1 Related work

As far as concerns the interaction modality, an immediate predecessor of *Harmonic Walk* is the *Stanza Logo-motora* [1], where the child has to use her/his spatial memory to match sounds with a spoken text.

Also the idea of exploiting the spatial properties of harmony is not new, as it has been widely developed in the *Harmony Space* project at the Music Computing Lab of the University of Stanford in 1993 [2]. The *Harmony Space* interface shows a desktop bi-dimensional matrix of pitches ordered by major third on the horizontal axis and by minor third on the vertical axis. Choosing a key area, a chord size and a chord mapping, when a note is selected, the chord built on it will sound. The interface has been used to simplify the study of harmony, to analyze musical pieces and to compose new ones. It exists also a larger physical version of the interface, which employs a floor projection of the harmonic grid and a camera tracking [3]. A more recent experience of interaction with Holland's *Harmony Space* has been proposed using a 3D graphical representation of the tonnetz. The user interacts with the interface through colored controllers which are used to select the pitches and to produce the audio feedback [4]. The main difference between *Harmonic Walk* and *Harmony Space* is that the latter fits a more expert user level. The environment is very rich and complex as it allows many option possibilities (the number of pitches used for the chord, the chord type, the key, etc.): so the interaction actually depends on a pre-selected series of options. Also the aim of the application is different. While *Harmony Space* has been conceived for the study and the analysis of music, *Harmonic Walk* invites the user to enter a physical tonnetz and to see what s/he can discover about it, employing a very simple and immediate interaction modality. It emphasizes the harmony space exploration rather than a real knowledge of harmonic rules, and for this reason is principally aimed at unexperienced users.

A GPS system to navigate a wide tonnetz area has also been proposed by Behringer and Elliott [5], where the authors suggest some musical games about composition and harmony features knowledge. However, a system spread on a surface of many square meters, involves an interaction modality completely different from that employed in the *Harmonic Walk*.

2. THE HARMONIC WALK'S THEORETICAL BACKGROUND

2.1 The harmonic space

The modern tonnetz, (the word literally means “web of tones”) is an efficient and popular way to represent the harmonic space. In the tonnetz, pitches are disposed along horizontal and diagonal lines. As shown in Fig. 2, the horizontal lines connect the circle of fifths (in red). On the diagonal axis SE-NW lays the major thirds circle (in blue), while the diagonal axis SW-NE bears the minor thirds circle (in yellow).

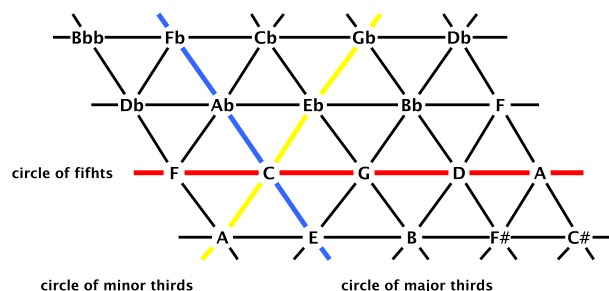


Figure 2. The modern tonnetz with the fifths, major and minor thirds axis.

Brian Hyer [6] developed an extensive theory of triadic transformations formalizing the so called neo-Riemannian chord operations PAR, REL and LT (leittonwechsel). With respect to the C minor chord the first inversion is around its lower edge, the CG axis (Fig. 3a). The result is the C major chord, defined by the lower triangle C-G-E. This is called “P transformation”, because it produces the parallel major chord of a minor chord. Inverting the same triangle chord around its right edge, causes a “R transformation” (REL), because the produced chord is the major relative Eb (Fig. 3b.) The same operation on the left side is a “L transformation” (LT, leading tone change) and produces the Ab chord (Fig. 3c). The whole group of these transformations is called “PRL transformations”. PRL transformations introduce the idea of parsimony in harmonic progressions. A chord progression is parsimonious if the two chords have at least two pitches in common [7]. From the geometric point of view this is expressed by the commonality of one edge of the two triangles, as can be seen in Fig. 4. Here the starting chord is C major and the PRL transformations are represented by the black arrows. But, as a matter of facts, the geometric form of the triangle allows another kind of transformation that is a vertex inversion. This transformation implies the existence of only one common note between the two chords. The vertex transformation enlarges the idea of parsimony [8], [9], including a greater number of chords with a different grade of proximity (the grey dashed arrows in Fig. 4). Accepting this further range of inversions we obtain a relatively coherent field of twelve chords: three by PRL edge transformations and nine by

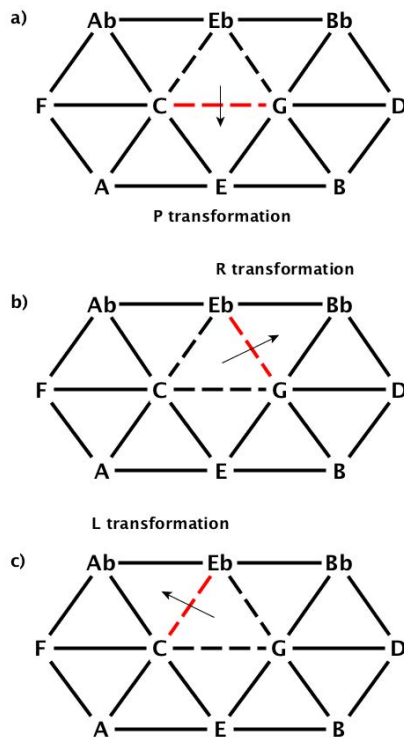


Figure 3. a) P transformation b) R transformation c) L transformation.

vertex transformations. To deliver a more precise measurement system of chords proximity degrees we introduce:

1. a pitch class definition of the harmonic relationships. Considering C as the starting point, we express the C major chord as the pitch class multiset $\{0, 4, 7\}$, where 0 is the unison, 4 is the major third and 7 is the fifth;
2. a displacement multiset associated with the harmonic relationship. E.g. the displacement multiset of the C major $\{0, 4, 7\}$ and c minor $\{0, 3, 7\}$ relationship is equal to $\{0, 1, 0\}$;
3. a smoothness value, as the sum of the members of the displacement multiset, which expresses how far the second chord moves from the first in semitones;
4. a parsimony degree $P_{m,n}$ where m expresses the number of moves by half step and n the number of moves by whole step.

As can be seen from Table 1, P and L transformations have the same smoothness and parsimony degree, while R has a smoothness of 2 and a parsimony degree $P_{0,1}$. This is due to the intrinsic asymmetry of the major and minor thirds axes along which the chords are overturned. The chords generated by the PRL and vertices inversions shape a rich harmonic world with traditional tonal harmonies, borrowed¹ and chromatic chords.

¹ Borrowed chords are commonly employed in modern as well as in 19th century harmony. They belong to parallel keys and can be used both in major and in minor mode.

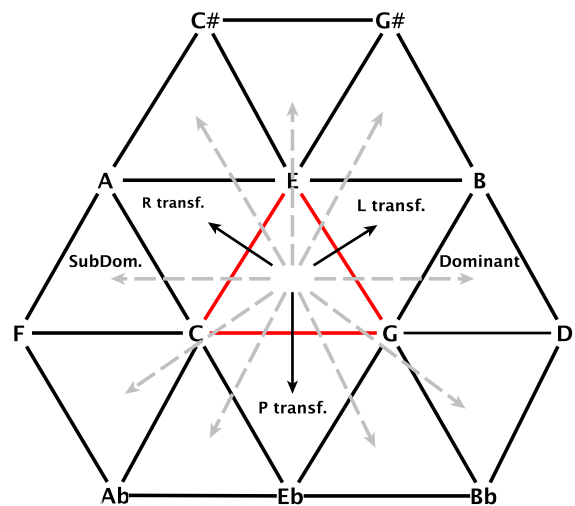


Figure 4. The tonnetz representation of PRL and vertices transformations with respect to the C major chord.

2.2 The physical space

The *Harmonic Walk* user's interface consists of a rectangular carpet put on the floor. It depicts a surface that comprehends the mapped space and that yields a useful visual reference for the perception of the centre and of the sides of the interface. The interaction happens when the user enters the sensorized zone and, walking on the carpet's bi-dimensional surface, moves on. The accessibility directions depend both on body's and world's asymmetries. Body's asymmetries hinge on the fact that, according to the spatial framework theory [10], we perceive the space around us by dividing it along the six sides of the body: head, feet, front, back, left and right (see Fig. 5).

The head-feet axis is the only one really asymmetric because it works only upward. The front-back axis is potentially symmetric, but is biased by the difficulty of perception and action in the world behind the back. Actually, the front-back axis separates the world in two parts: the one in front which can be easily reached, and the one on the rear which cannot even be seen. West and east lay on the left-right axis, which is symmetric: this means that these two directions could be equivalent.

As far as concerns world asymmetries, they can depend on a lot of elements like the room's door and windows orientation, the direction of the light, the other people's positions, the way the furniture is disposed and so on. In particular a very strong constraint could concern how the sensorized area is located inside the room. The door position and the proximity of one or more sides of the application's surface to the room walls, may be a strong spatial bias for the users when they decide to enter the application, influencing the number of available access possibilities. Another important asymmetry to be considered is the position of the teacher who guides the child in his environmental exploration and completion tasks. As on the surface there is

Chords	Pitch classes	Transformation	Harmonic relationship	Displacement Multiset	Smoothness	Proximity
C major c minor	{0, 4, 7} {0, 3, 7}	P (low edge inversion)	Minor Parallel	{0, 1, 0}	1	$P_{1,0}$
C major e minor	{4, 7, 0} {4, 7, 11}	L (right edge inversion)	Mediant	{0, 0, 1}	1	$P_{1,0}$
C major a minor	{4, 7, 0} {4, 9, 0}	R (left edge inversion)	Minor Relative	{0, 2, 0}	2	$P_{0,1}$
C major A major	{0, 4, 7} {1, 4, 9}	Upper vertex inversion	Major Superdominant	{1, 0, 2}	3	$P_{1,1}$
C major c# minor	{0, 4, 7} {1, 4, 8}	Upper vertex inversion	Chromatic relationship	{1, 0, 1}	2	$P_{2,0}$
C major E major	{4, 7, 0} {4, 8, 11}	Upper vertex inversion	Major Mediant	{0, 1, 1}	2	$P_{2,0}$
C major F major	{0, 4, 7} {0, 5, 9}	Left vertex inversion	Subdominant	{0, 1, 2}	3	$P_{1,1}$
C major f minor	{0, 4, 7} {0, 5, 8}	Left vertex inversion	Minor Subdominant	{0, 1, 1}	2	$P_{2,0}$
C major Ab major	{0, 4, 7} {0, 3, 8}	Left vertex inversion	Flat sixth	{0, 1, 1}	2	$P_{2,0}$
C major G major	{4, 7, 0} {2, 7, 11}	Right vertex inversion	Dominant	{2, 0, 1}	3	$P_{1,1}$
C major Eb major	{4, 7, 0} {3, 7, 10}	Right vertex inversion	Flat third	{1, 0, 2}	3	$P_{1,1}$
C major g minor	{4, 7, 0} {2, 7, 10}	Right vertex inversion	Minor Dominant	{2, 0, 2}	4	$P_{2,2}$

Table 1. The PRL and vertices transformations chord’s table with respect to the C major chord.

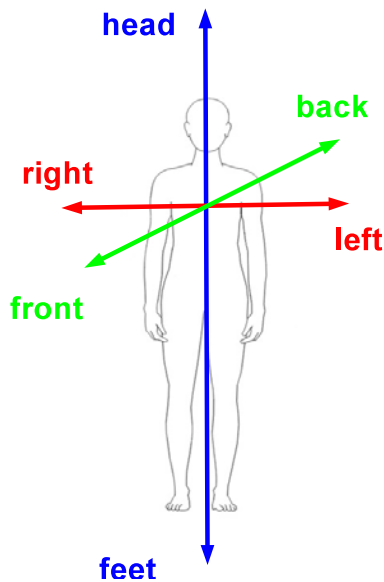


Figure 5. The human body axes, illustrating the head-feet, front-back and left-right relative directions.

no visible referring point, we can assume that the teacher’s side makes the user’s orientation no longer neutral and that they will probably perceive this direction as the most important with respect to the others. A last very important bias is the rectangular form of the application surface itself, where sides and centre can play an important role in the user’s spatial perception.

We describe the orientation of the application in Fig. 6, where we use cardinal points or, more easily, the clock metaphor. By convention, we put the south (or h. 6) in the direction of the teacher and, consequently, we name all the other directions.

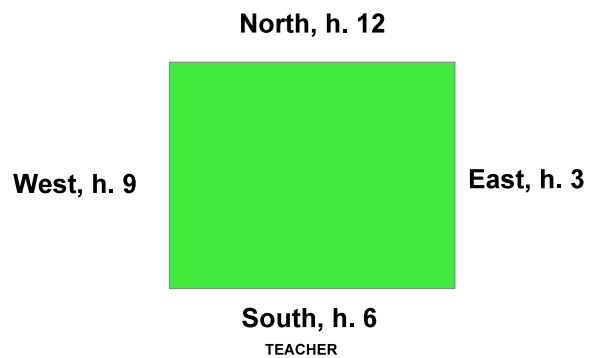


Figure 6. The orientation of the application’s interface.

3. THE “HARMONIC WALK” APPLICATION

3.1 System architecture

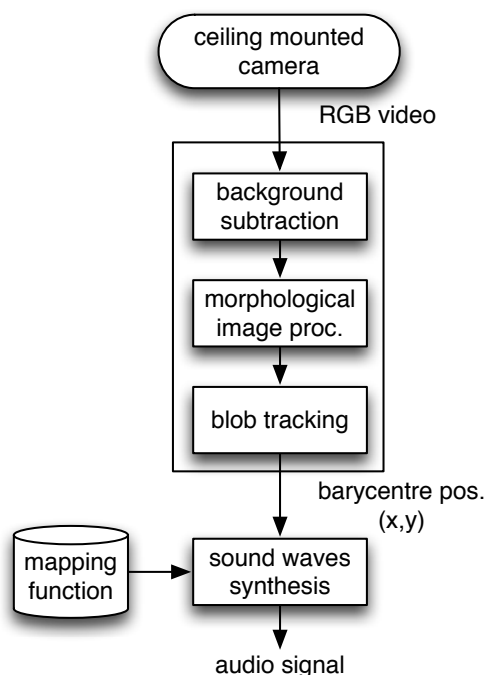


Figure 7. The system architecture.

The *Harmonic Walk*² architecture is composed by two software modules, aimed at video analysis and sound synthesis respectively (see Fig. 7). A ceiling mounted video camera, oriented perpendicularly to the floor, captures the users' movements inside a rectangular area, whose dimensions depend both from the distance camera-floor and the field of view of the lens. As the system is designed for carrying out educational activities inside a classroom, the camera lens is chosen in order to view a rectangle of about 3x4 meters when the camera is mounted on the ceiling of the room. The video module analyzes the input images in three steps: first, the background is subtracted following the averaging background method proposed in [11], with adaptive thresholds for each color channel; then, the resulting black and white images are processed by means of morphological transformations [12] in order to obtain well shaped blobs, representing the users' silhouettes seen from the top; finally, the blobs moves are tracked and the two-dimensional barycenter of each blob is calculated. Although the video analysis module can track more users simultaneously, this first implementation of the *Harmonic Walk* is designed to be used by a single user at a time. The user's barycenter coordinates are sent via OSC to the sound synthesis module, implemented in the Max/MSP environment. Depending on the user's position, the sound module provides an audio output, composed by a particular harmonic chord, following a pre-defined map (see Fig. 10). Each chord is synthesized using four different wave shapes mixed together to form a uniform sound. A proper enve-

² See <http://smc.dei.unipd.it/harmonicwalk.html> for the *Harmonic Walk* videos, Max/Msp patches and sound samples.

lope and reverberation effect ensures a certain persistence of the sound also after the sounding zone has been hit. As soon as the sounding zone is abandoned, the reverberation effect is muted, stopping any residual audio feedback.

3.2 The Harmonic Walk's Cognitive Map

As soon as users enter the application's surface, they begin to employ their previously acquired spatial learning, one of whose fundamental elements are landmarks knowledge. Landmarks are important because they are the milestones of the route [13]. At the beginning the only thing the children know is that they have to find four sounds. But they have no idea of what kind of sounds will be discovered and where these sounds are. So they will begin to move in the search employing locomotion, which refers to the guidance of oneself through space. In this initial condition, locomotion is a free wandering, as the children haven't yet any available data to build a cognitive map.

As soon as the children discover the first sounding area, they register the first landmark, beginning so to feed their application's cognitive map. A spatial cognitive map is constituted by many stored discrete pieces of knowledge including landmarks, route segments and regions. In the case of our application, the landmarks are represented by the chords that are set on the surface of the carpet; the regions are represented by the areas corresponding to a determined chord; the route segments are represented by the paths that the user employs to link the various chords. Nevertheless, a spatial cognitive map is much more than a mental routing sketch, as it includes other non-spatial elements, like perceptual attributes and emotions. In the case of the *Harmonic Walk*, the sound perceptual attribute itself is the landmark, and the various landmarks are differentiated in base of their position and in base of the fact that these attributes are perceived as different. The discovery of a second sounding zone adds another landmark to the children's map, and so on. When the exploration phase is completed, we ask the children to try to connect the found areas to see if the harmonic progressions played are of some interest for them. This involves a certain number of abilities that are built in various subsequent stages:

1. to distinguish a chord from another,
2. to locate them and to maintain a steady position so to listen to the new found chord,
3. to remember the regions where the various chords are,
4. to provide a route to reach the various stored regions,
5. to navigate the discovered sounding regions in search of some convincing harmonic progression.
6. When one is found, the ability to build one or more navigation patterns linking the various chords and, at the end,
7. to perform the chosen pattern repeatedly.

For each of these stages we provide an evaluation grid to be used by the observers during the test.

4. EXPERIMENTAL ASSESSMENT

4.1 Subjects

We organize a series of observational tests to collect some raw data from the potential users of the *Harmonic Walk* application.

We identify our primary users in upper elementary school students, approximately from 7 to 11 years old (the second cycle of Italian primary school). The tests carried on with the *Stanza Logo-motora* [1], whose interface is similar to the *Harmonic Walk*, showed clearly that children represent a privileged target for such applications, as the interaction with the interface is very simple and it seems like a very amusing game for a child. As the cognitive content of the *Harmonic Walk* is not trivial, we advise at least an upper elementary user's level.

4.2 Materials

To perform the tests we divide the interface's surface in nine regions. We also choose to employ only one spatial schema: an inverted "T". In this disposition we have a peripheral displacement, a center and a right/left choice. If we superimpose the "T" schema to the nine interface's regions and put the centre of the "T" on the centre of the application's surface, we obtain the spatial schema showed in Fig. 8, where only 4 zones are filled with sound, while

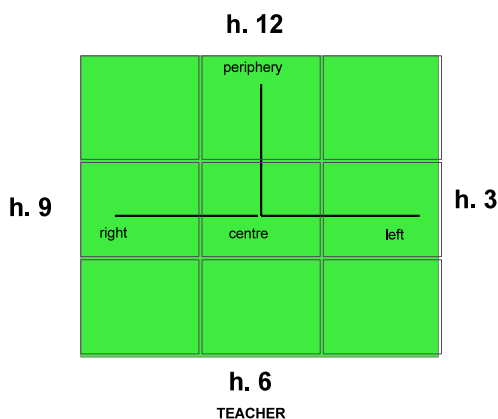


Figure 8. The "T" schema in the nine zones of the interface.

all the others are left empty.

Applying the "T" disposition to the tonnetz, we can choose between the following two chords progressions:

1. c# minor (periphery), C major (centre), F and G major (R/L choices), in red in Fig. 9;
2. C major (periphery), c minor (centre), Ab and Eb major (R/L choices), in blue in Fig. 9.

We name the first red schema "T far" because the chords are all generated by vertices transformations; the second blue schema instead is "T near", as here all the chords are generated by PRL transformations. Moreover in the "T far" schema the horizontal lines link F-C and G major

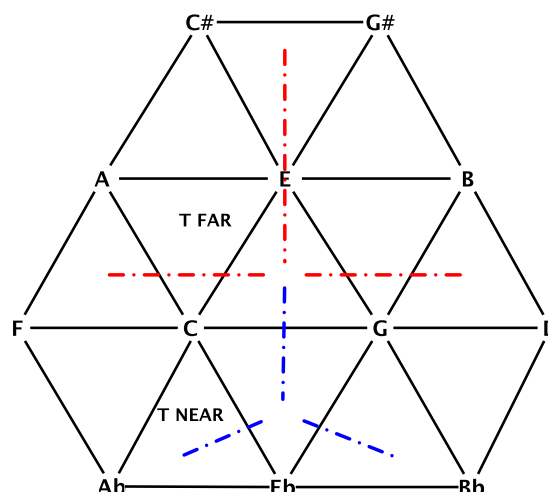


Figure 9. The "T" schema related to the tonnetz with two different harmonic progressions.

which are respectively the subdominant, tonic and dominant chords of the tonality of C major. This path has a very strong perceptual weight, while the peripheral chord c# minor, also if produced by the same kind of transformation and probably owing to its chromatic relationship with the C major, sounds very far apart from the other three chords. The "T near" schema, on the contrary, shows much softer and homogeneous relationships, as the changes among its chords have a subtler perceptual quality. In Fig. 10 is represented the final schema of the mapping used for the test, with the two chords progressions to be used alternately.

4.3 Procedure

The test is carried one child at a time. The position of the carpet is in the middle of a large room, while the door is in the western side; the teacher seats in the southern side of the application's surface. The teacher will assist the children during the test, assigning them an ordered list of subsequent tasks. Only the teacher will speak to the children during the test, while all the other observers will be silent. The teacher tells the children that four sounding zones are hidden on the carpet. He/she asks them :

1. to explore the carpet in search of the four zones corresponding to the four different chords (zone discovery);
2. when at least two are found, to detect a first path between the found zones and to perform it repeatedly;
3. if a first path is successfully found, to detect a second path;
4. to choose one preferred path among the two and to perform it repeatedly (discrimination).

The children perform their tasks without the help of any visual cue, relying only on the audio feedback and on their

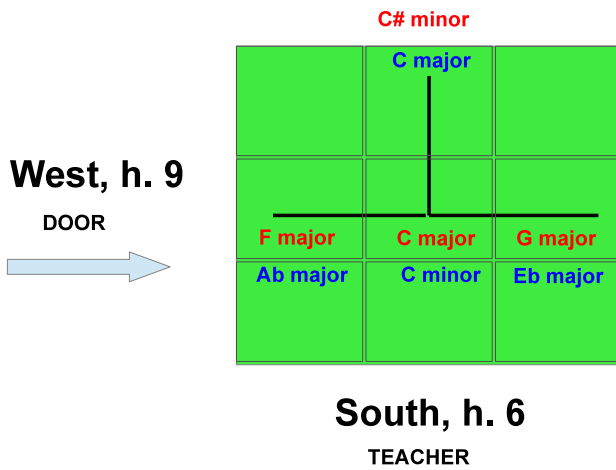


Figure 10. The map of the space used for the experimental tests with the “T far” schema (in red) and the “T near” schema (in blue) with the door position.

spatial and orientation skills. The grid presented in Fig. 10 is explanatory and is not reported on the carpet. In this way the children are left free to build a path as they like, also going through silent zones.

4.4 Results

Table 2 reports the results of the tested group’s successes as far as concerns the completion of the assigned tasks.

Tasks	Zone Disc.	1st Path	2nd Path	Discr.
N. of cdn.	15	19	14	12
Perc.	75%	95%	70%	60%

Table 2. Successes obtained in pursuing the test’s four tasks.

The great majority of the children (75%) were able to locate and to remember the position of the four chords they had to find on the carpet. Nearly all of them (95%), also if not able to find and to remember the position of all the four chords, were able to find a path linking at least two chords. As the tasks become more and more difficult, we see that the number of successes decreases. Actually, only the 70% of the children could find a second alternative path, and only the 60% were able to remember clearly the two different chord progressions so generated. Not all the children could express a preference among the two paths, also if they were able to discriminate the two harmonic progressions. Moreover, nearly no child was able to explain why they had preferred a determined path nor we noted differences between the group tested with the “T far” as with the “T near” chord progression.

A second observation set concerned the influence of world’s asymmetries elements like the door and the teacher position and the form of the application surface. In particular we observed the starting point of the paths the children selected after the exploration phase.

Among 33 recorded paths (see Table 3), 11 had the centre

Starting point	Centre	h. 9	h. 3	h. 12
N. of choices	11	11	7	4
Percentage	33%	33%	21%	12%

Table 3. Preferred path starting points.

as a starting point, 11 the h. 9 position, 7 the h. 3 and 4 the h. 12. In this case the numbers confirm the strength of the world’s asymmetries. In a rectangular form the centre is perceived as the only one point of equilibrium, and this explains one third of the choices. The other third is probably due to the door position, which marks the western side of the application (h. 9) as the most important side, influencing so the choice of the path’s starting point. We also can observe that the door position bias was much stronger than the teacher’s position, which would have led to choose the h. 12 position as the preferred starting point.

5. CONCLUSIONS AND FURTHER WORK

In this paper we described a system which allows an unskilled user to interact with the world of harmony in a simple and immediate way. A preliminary experiment was carried to show that children can locate different chords scattered on the application surface, remember their position and find one or more paths to link them. These fundamental premises are crucial for the development of the application, as they constitute the basilar conditions for the user’s interaction. In our experimental tests world’s asymmetries proved to play an important role, as they determined the behavior of the majority of the children, biasing the choice of their paths starting points. This interaction aspect needs further investigation also in relation to the spatial disposition of the chords and to their perceptual qualities. As far as concerns body’s asymmetries, it can be noticed that the only directional axis employed for motion was always the front. It seems that the search for the target position was so compelling for the children, not to allow them any other motion direction.

The *Harmonic Walk* has proved to be enough robust to be used in many different application fields. The first, obvious one, is the music didactics with many possible projects to develop, like studies on different harmonic systems, voice intonation, melody accompaniment, improvisation and composition. Also if the tonnetz harmonic space representation has a very strong theoretical background, it can turn out to be unfit for tonal music applications, where the possibility of a direct link from a tonic chord to different many other chord degrees could be necessary. So, we might need to discuss and to experiment also other spatial chord dispositions. Moreover, as spatial thinking has been recognized by the educationists to have a very strong power of conveying information [14], the same spatial concepts and interaction modalities employed for the *Harmonic Walk* could be shared by other applications designed for the study of mathematics, geography, geometry or science. But, beyond these practical applications, more general research trends can be found to develop the *Har-*

monic Walk's interface. The first one is to increase the interactivity of the system, tracking not only the blob, but also some other parts of the body, for instance the hands position. The second is to add visual cues to help orientation, like for instances floor projections. This could help the users to concentrate towards audio or video feedback generated by their interaction with the environment. This perspective leads to the creation of a widespread, fully immersive artificial environment where audio and video feedback could guide the user towards knowledge in a more complete, easy and satisfying way.

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